Large N Partition Functions, Holography, and Black Holes

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The large N team

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Motivation

Supersymmetric localization allows for the exact calculation of physical observables in supersymmetric QFTs.

Apply this tool to SCFTs with holographic duals in string and M-theory.

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Goal: Describe recent progress on these topics for 3d SCFTs with AdS_4 duals in M-theory.

Plan

- Motivation ✓
- The ABJM theory on S^3
- The ABJM topologically twisted index
- Holography and black holes
- Other 3d $\mathcal{N} = 2$ SCFTs
- The superconformal index
- Thermal observables

The ABJM theory on S^3

ABJM and holography

The ABJM theory: $\mathrm{U}(N)_k \times \mathrm{U}(N)_{-k}$ CS-matter theory with two pairs of bi-fundamental chirals $(A_{1,2}, B_{1,2})$ and superpotential

$$\mathcal{W} = \mathrm{Tr}(A_1 B_1 A_2 B_2 - A_1 B_2 A_2 B_1)\,.$$

For $k > 2$ it has $\mathcal{N} = 6$ supersymmetry and $\mathrm{SU}(4)_R \times \mathrm{U}(1)_b$ global symmetry. Describes N M2-branes on $\mathbb{C}^4/\mathbb{Z}_k$.

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- In the limit of fixed k and large N, the ABJM theory is dual to the M-theory background $\mathrm{AdS}_4 \times S^7/\mathbb{Z}_k$

$$(L/\ell_{\mathrm{P}})^6 \sim k\,N\,.$$

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$$(L/\ell_{\mathrm{P}})^6 \sim k\,N\,.$$

- At large k and fixed 't Hooft coupling $\lambda = N/k$ the theory is dual to type IIA string theory on $\mathrm{AdS}_4 \times \mathbb{CP}^3$

$$k\,g_{\mathrm{st}} = L/\ell_{\mathrm{s}} \sim \lambda^{1/4}\,.$$

  Perturbative type IIA string theory at large k and small g_{st}, i.e. fixed λ and large N.

ABJM on S^3

The path integral on S^3 can be computed by supersymmetric localization and reduces to a matrix model [Kapustin-Willett-Yaakov]

$$Z(N,k) = \frac{1}{N!^2} \int \frac{d^N\mu}{(2\pi)^N} \frac{d^N\nu}{(2\pi)^N} \, \exp\left[\frac{\mathrm{i}k}{4\pi} \sum_{i=1}^{N} (\mu_i^2 - \nu_i^2)\right] \frac{\Pi_{i<j} \left[2\sinh(\frac{\mu_i-\mu_j}{2})\right]^2 \left[2\sinh(\frac{\nu_i-\nu_j}{2})\right]^2}{\Pi_{i,j} \left[2\cosh(\frac{\mu_i-\nu_j}{2})\right]^2}$$

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Three methods have been used to study $Z(N,k)$ at large N

1. Map to CS theory on $S^3/\mathbb{Z}_2$ (or topological strings on $\mathbb{P}^1 \times \mathbb{P}^1$) and solve with large N techniques. Applies at large N, fixed N/k.[Drukker-Mariño-Putrov]
2. Study the large N limit at fixed k numerically.[Herzog-Klebanov-Pufu-Tesileanu]
3. Map the problem to a free Fermi gas on the real line with non-standard kinetic term. Valid at large N and finite k.[Mariño-Putrov]

ABJM at large N - An Airy tale

At large N and fixed k the S^3 partition function of the ABJM theory is [Mariño-Putrov], [Fuji-Hirano-Moriyama]

$$Z_{S^3} = e^{\mathcal{A}(k)} C^{-\frac{1}{3}} \mathrm{Ai}[C^{-\frac{1}{3}}(N-B)] + \mathcal{O}(e^{-\sqrt{N}}) \, ,$$

with

$$B = \frac{k}{24} + \frac{1}{3k} \, , \qquad C = \frac{2}{\pi^2 k} \, ,$$

and

$$\mathcal{A}(k) = \frac{2\zeta(3)}{\pi^2 k}\left(1 - \frac{k^3}{16}\right) + \frac{k^2}{\pi^2}\int_0^\infty \frac{x \log\left(1 - e^{-2x}\right)}{e^{kx}-1} \, dx$$
$$= -\frac{\zeta(3)}{8\pi^2} k^2 + 2\zeta'(-1) + \frac{1}{6}\log\frac{4\pi}{k} + \sum_{n\geq 2}\left(\frac{2\pi}{k}\right)^{2n-2} \frac{(-4)^{n-1} B_{2n} B_{2n-2}}{n(2n-2)(2n-2)!} \, .$$

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The large N expansion takes the explicit form

$$-\log Z_{S^3} = \frac{2}{3\sqrt{C}} N^{\frac{3}{2}} - \frac{B}{\sqrt{C}} N^{\frac{1}{2}} + \frac{1}{4}\log N - \mathcal{A}(k) + \frac{1}{4}\log\frac{32}{k} + \mathcal{O}(N^{-\frac{1}{2}}) \,.$$

ABJM at large N - An Airy tale

This can be reorganized *à la 't Hooft* into a type IIA string theory expansion

$$F_{S^3} = -\sum_{\mathrm{g}\geq 0} (2\pi \mathrm{i}\lambda)^{2\mathrm{g}-2} F_{\mathrm{g}}(\lambda)\, N^{2-2\mathrm{g}}\,.$$

The genus g type IIA free energies can be computed systematically (up to $e^{-\sqrt{\lambda}}$ corrections) and read (agrees with topological string results)

$$\begin{aligned}
F_0(\lambda) &= \frac{4\pi^3\sqrt{2}}{3}\,\hat{\lambda}^{\frac{3}{2}} + \frac{\zeta(3)}{2}\,,\\
F_1(\lambda) &= \frac{\pi}{3\sqrt{2}}\,\hat{\lambda}^{\frac{1}{2}} - \frac{1}{4}\log\hat{\lambda} + \frac{1}{6}\log\lambda + \frac{1}{12}\log\frac{\pi^2}{32} + 2\zeta'(-1) - \frac{1}{2}\log 2\,,\\
F_2(\lambda) &= \frac{5\,\hat{\lambda}^{-\frac{3}{2}}}{96\pi^3\sqrt{2}} - \frac{\hat{\lambda}^{-1}}{48\pi^2} + \frac{\hat{\lambda}^{-\frac{1}{2}}}{144\pi\sqrt{2}} - \frac{1}{360}\,,\\
F_3(\lambda) &= \frac{5\,\hat{\lambda}^{-3}}{512\pi^6} - \frac{5\,\hat{\lambda}^{-\frac{5}{2}}}{768\pi^5\sqrt{2}} + \frac{\hat{\lambda}^{-2}}{1152\pi^4} - \frac{\hat{\lambda}^{-\frac{3}{2}}}{10368\pi^3\sqrt{2}} - \frac{1}{22680}\,,
\end{aligned}$$

where

$$\hat{\lambda} = \lambda - \frac{1}{24}\,.$$

ABJM on S^3 and holography

These results are prime targets for string/M-theory and AdS_4 holography!

$$-\log Z_{S^3} = \frac{\pi\sqrt{2k}}{3}\, N^{\frac{3}{2}} - \frac{\pi}{3\sqrt{2}} \left(\frac{1}{8}k^{\frac{3}{2}} + k^{-\frac{1}{2}}\right) N^{\frac{1}{2}} + \frac{1}{4}\log N + \dots .$$

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From the on-shell action of the $\mathrm{AdS}_4 \times X_7$ supergravity solution one finds

$$F_{S^3} = -\log Z_{S^3} = \sqrt{\frac{2\pi^6}{27\mathrm{vol}(X_7)}} N^{\frac{3}{2}} .$$

Plug in $\mathrm{vol}(S^7/\mathbb{Z}_k) = \frac{\pi^4}{3k}$ to find a match with the localization result.

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The $N^{\frac{1}{2}}$ term can be derived using the on-shell action of four-derivative gauged supergravity. [NPB-Charles-Hristov-Reys]

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The $\log N$ term is obtained by a 1-loop calculation of the 11d supergravity fields around AdS$_4 \times S^7$.[Bhattacharyya-Grassi-Mariño-Sen]

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Note: Derive the full Airy function using supersymmetric localization in supergravity on AdS_4? [Dabholkar-Drukker-Gomes]

Adding sources

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The ABJM S^3 partition function with a $\mathrm{U}(1)\times\mathrm{U}(1)$ invariant squashing and real mass deformation takes the form [NPB-Hong-Reys], [Nosaka], [Hatsuda], [Hristov], [Chester-Kalloor-Sharon], [Minahan-Naseer-Thull]

$$Z_{S^3}(N,k,\Delta,b) = e^{\mathcal{A}(k,\Delta,b)} C_k^{-\frac{1}{3}} \mathrm{Ai}[C_k^{-\frac{1}{3}}(N-B_k)] + \mathcal{O}(e^{-\sqrt{N}})$$

with

$$C_k = \frac{2}{\pi^2 k} \frac{(b+b^{-1})^{-4}}{\prod_{a=1}^4 \Delta_a}\,, \quad B_k = \frac{k}{24} - \frac{1}{12k}\sum_{a=1}^4 \frac{1}{\Delta_a} + \frac{1-\frac{1}{4}\sum_a \Delta_a^2}{3k(b+b^{-1})^2\prod_{a=1}^4 \Delta_a}\,,$$

and

$$\Delta_1 = \frac{1}{2} - \mathrm{i}\,\frac{m_1+m_2+m_3}{b+b^{-1}}\,,\ \Delta_2 = \frac{1}{2} - \mathrm{i}\,\frac{m_1-m_2-m_3}{b+b^{-1}}\,,$$
$$\Delta_3 = \frac{1}{2} + \mathrm{i}\,\frac{m_1+m_2-m_3}{b+b^{-1}}\,,\ \Delta_4 = \frac{1}{2} + \mathrm{i}\,\frac{m_1-m_2+m_3}{b+b^{-1}}\,,$$

such that $\sum_a \Delta_a = 2$.

Adding sources

This result encodes integrated correlation functions of the ABJM theory on $\mathbb{R}^3$.

Expand at large N and use holography to constrain/compute the higher-derivative corrections to type II string theory and M-theory. [Chester-Pufu-Yin], [Binder-Chester-Pufu], ...

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Example:

Squashed S^3 partition function

$$F_{S_b^3} = \frac{\pi\sqrt{2k}}{12}\left[(b+b^{-1})\left[N^{\frac{3}{2}} + \left(\frac{1}{k} - \frac{k}{16}\right)N^{\frac{1}{2}}\right] - \frac{6}{k}N^{\frac{1}{2}}\right] + \frac{1}{4}\log N + \mathcal{O}(N^0)\,.$$

This captures integrated correlators of $T_{\mu\nu}$ for the ABJM theory in flat space!

$$C_T = \frac{32}{\pi^2}\left(\frac{\partial^2 F_{S_b^3}}{\partial b^2}\right)_{b=1} = \frac{64\sqrt{2k}}{3\pi}N^{\frac{3}{2}} + \frac{4(16-k^2)\sqrt{2}}{3\pi\sqrt{k}}N^{\frac{1}{2}} + \mathcal{O}(N^0)\,.$$

where

$$\langle T_{\mu\nu}T_{\rho\sigma}\rangle = \frac{C_T}{(48\pi)^2}\left(P_{\mu\rho}P_{\nu\sigma} + P_{\nu\rho}P_{\mu\sigma} - P_{\mu\nu}P_{\rho\sigma}\right)\frac{1}{\vec{x}^2}\,, \quad P_{\mu\nu} \equiv \delta_{\mu\nu}\partial^2 - \partial_\mu\partial_\nu\,.$$

What about the ABJM partition function on other 3-manifolds?

The ABJM topologically twisted index

TTI

The topologically twisted index (TTI) is a partition function of a 3d $\mathcal{N} = 2$ SCFT on $S^1 \times \Sigma_{\mathfrak{g}}$. Supersymmetry is preserved by Witten's topological twist on $\Sigma_{\mathfrak{g}}$. The 3d QFT is not topological. Using supersymmetric localization the path integral can be reduced to a matrix integral. [Benini-Zaffaroni], [Closset-Kim]

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For the ABJM theory the result is

$$
\begin{aligned}
Z_{S^1\times\Sigma_{\mathfrak{g}}}(N,k,\Delta,\mathfrak{n}) = {} & \frac{1}{(N!)^2}\sum_{\mathfrak{m},\tilde{\mathfrak{m}}\in\mathbb{Z}_N}\oint_{\mathcal{C}}\prod_{i=1}^{N}\frac{dx_i}{2\pi\mathrm{i}x_i}\prod_{j=1}^{N}\frac{d\tilde{x}_j}{2\pi\mathrm{i}\tilde{x}_j}\prod_{i=1}^{N}x_i^{k\mathfrak{m}_i}\prod_{j=1}^{N}\tilde{x}_j^{-k\tilde{\mathfrak{m}}_j} \\
& \times\left(\det\mathbb{B}(N,k,x,\tilde{x},\Delta)\right)^{\mathfrak{g}}\times\prod_{i\neq j}^{N}\left(1-\frac{x_i}{x_j}\right)^{1-\mathfrak{g}}\left(1-\frac{\tilde{x}_i}{\tilde{x}_j}\right)^{1-\mathfrak{g}} \\
& \times\prod_{i,j=1}^{N}\prod_{a=1,2}\left(\frac{\sqrt{\frac{x_i}{\tilde{x}_j}y_a}}{1-\frac{x_i}{\tilde{x}_j}y_a}\right)^{\mathfrak{m}_i-\tilde{\mathfrak{m}}_j+1-\mathfrak{g}-\mathfrak{n}_a}\prod_{a=3,4}\left(\frac{\sqrt{\frac{\tilde{x}_j}{x_i}y_a}}{1-\frac{\tilde{x}_j}{x_i}y_a}\right)^{-\mathfrak{m}_i+\tilde{\mathfrak{m}}_j+1-\mathfrak{g}-\mathfrak{n}_a} .
\end{aligned}
$$

Here $y_a = e^{\mathrm{i}\pi\Delta_a}$ and supersymmetry imposes

$$
\sum_{a=1}^{4}\Delta_a = 2\,, \qquad \sum_{a=1}^{4}\mathfrak{n}_a = 2(1-\mathfrak{g})\,.
$$

TTI

Using (subtle) contour integration the TTI can be rewritten as

$$Z = \prod_{a=1}^{4} y_a^{-\frac{N^2}{2}\mathfrak{n}_a} \sum_{\{x_i, \tilde{x}_j\}} \left[\frac{1}{\det \mathbb{B}} \frac{\prod_{i=1}^{N} x_i^N \tilde{x}_i^N \prod_{i \neq j}^{N} (1 - \frac{x_i}{x_j})(1 - \frac{\tilde{x}_i}{\tilde{x}_j})}{\prod_{i,j=1}^{N} \prod_{a=1}^{2} (\tilde{x}_j - x_i y_a)^{1 - \frac{\mathfrak{n}_a}{1-\mathfrak{g}}} \prod_{a=3}^{4} (x_i - \tilde{x}_j y_a)^{1 - \frac{\mathfrak{n}_a}{1-\mathfrak{g}}}} \right]^{1-\mathfrak{g}}$$

Where x_i and $\tilde{x}_i$ are solutions to the following "Bethe Ansatz Equations"

$$e^{\mathrm{i}B_i} \equiv x_i^k \prod_{j=1}^{N} \frac{(1 - y_3 \frac{\tilde{x}_j}{x_i})(1 - y_4 \frac{\tilde{x}_j}{x_i})}{(1 - y_1^{-1} \frac{\tilde{x}_j}{x_i})(1 - y_2^{-1} \frac{\tilde{x}_j}{x_i})} = 1\,,$$

$$e^{\mathrm{i}\tilde{B}_j} \equiv \tilde{x}_j^k \prod_{i=1}^{N} \frac{(1 - y_3 \frac{\tilde{x}_j}{x_i})(1 - y_4 \frac{\tilde{x}_j}{x_i})}{(1 - y_1^{-1} \frac{\tilde{x}_j}{x_i})(1 - y_2^{-1} \frac{\tilde{x}_j}{x_i})} = 1\,,$$

and the Jacobian matrix $\mathbb{B}$ is given by

$$\mathbb{B} = \frac{\partial(e^{\mathrm{i}B_1}, \cdots, e^{\mathrm{i}B_N}, e^{\mathrm{i}\tilde{B}_1}, \cdots, e^{\mathrm{i}\tilde{B}_N})}{\partial(\log x_1, \cdots, \log x_N, \log \tilde{x}_1, \cdots, \log \tilde{x}_N)}\,.$$

TTI at large N

The BAE solution in the large N limit takes the form [Benini-Hristov-Zaffaroni]

$$\log x_i = N^{\frac{1}{2}}\, t_i - \mathrm{i} v_i\,, \quad \log \tilde{x}_j = N^{\frac{1}{2}}\, t_j - \mathrm{i}\tilde{v}_j\,.$$

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Our approach: Use this solution as a starting point to numerically solve the BAE and calculate the index. The numerical results are very precise and led us to an analytic form for the TTI valid to all orders in $1/N$!

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To write the result compactly define

$$\hat{N}_\Delta \equiv N - \frac{k}{24} + \frac{1}{12k}\sum_{a=1}^{4}\frac{1}{\Delta_a}\,,$$

in terms of which $F_{S^1\times\Sigma_\mathfrak{g}} = -\log Z_{S^1\times\Sigma_\mathfrak{g}}$, takes the simple form:

$$F_{S^1\times\Sigma_\mathfrak{g}} = \frac{\pi\sqrt{2k\Delta_1\Delta_2\Delta_3\Delta_4}}{3}\sum_{a=1}^{4}\frac{\mathfrak{n}_a}{\Delta_a}\left(\hat{N}_\Delta^{\frac{3}{2}} - \frac{\mathfrak{c}_a}{k}\hat{N}_\Delta^{\frac{1}{2}}\right) + \frac{1-\mathfrak{g}}{2}\log\hat{N}_\Delta - \hat{f}_0(k,\Delta,\mathfrak{n})\,,$$

where $\mathfrak{c}_a$ are given by

$$\mathfrak{c}_a = \frac{\prod_{b\neq a}(\Delta_a+\Delta_b)}{8\Delta_1\Delta_2\Delta_3\Delta_4}\sum_{b\neq a}\Delta_b\,.$$

The universal index

The universal index is defined by setting $\Delta_a = \frac{1}{2}$ and $\mathfrak{n}_a = \frac{1-\mathfrak{g}}{2}$. We then define $\hat{N} = N - \frac{k}{24} + \frac{2}{3k}$ to obtain

$$\frac{F_{S^1 \times \Sigma_\mathfrak{g}}}{1-\mathfrak{g}} = \frac{\pi\sqrt{2k}}{3}\left(\hat{N}^{\frac{3}{2}} - \frac{3}{k}\hat{N}^{\frac{1}{2}}\right) + \frac{1}{2}\log\hat{N} - \hat{f}_0(k)\,.$$

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$$\frac{F_{S^1\times\Sigma_{\mathfrak{g}}}}{1-\mathfrak{g}} = \frac{\pi\sqrt{2k}}{3}\left(\hat{N}^{\frac{3}{2}} - \frac{3}{k}\hat{N}^{\frac{1}{2}}\right) + \frac{1}{2}\log\hat{N} - \hat{f}_0(k)\,.$$

No closed form expression for $\hat{f}_0(k)$ but at large k we find

$$\hat{f}_0(k) = -\frac{3\zeta(3)}{8\pi^2}k^2 + \frac{7}{6}\log k + \mathfrak{f}_0 + \sum_{n=1}^{5}\left(\frac{2\pi}{k}\right)^{2n}\frac{\mathfrak{f}_{2n}}{3^{n+2}} + \mathcal{O}(k^{-12})\,,$$

with $\{\mathfrak{f}_{2n}\} = \left\{-\frac{6}{5},\ \frac{19}{70},\ -\frac{41}{175},\ \frac{279}{700},\ -\frac{964636}{875875}\right\}$ and $\mathfrak{f}_0 = -2.096848299$.

The universal index

The universal index is defined by setting $\Delta_a = \frac{1}{2}$ and $\mathfrak{n}_a = \frac{1-\mathfrak{g}}{2}$. We then define $\hat{N} = N - \frac{k}{24} + \frac{2}{3k}$ to obtain

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For low values of k, $\hat{f}_0(k)$ can be determined numerically with very good precision:

$$\hat{f}_0(1) = -3.045951311\,,\quad \hat{f}_0(2) = -1.786597534\,,$$
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We have checked these results with extensive numerical calculations to great accuracy. They are exact up to $e^{-\sqrt{N}}$ corrections.

The universal index

This result can be reorganized as a type IIA string theory expansion

$$F_{S^1\times\Sigma_{\mathfrak{g}}} = -\sum_{g\geq 0} (2\pi\mathrm{i}\lambda)^{2g-2} F_g(\lambda)\, N^{2-2g}\,.$$

For low genera we find

$$\begin{aligned}
\frac{F_0(\lambda)}{1-\mathfrak{g}} &= \frac{4\pi^3\sqrt{2}}{3}\,\hat{\lambda}^{\frac{3}{2}} + \frac{3\zeta(3)}{2}\,,\\
\frac{F_1(\lambda)}{1-\mathfrak{g}} &= \frac{2\pi\sqrt{2}}{3}\,\hat{\lambda}^{\frac{1}{2}} - \frac{1}{2}\log\hat{\lambda} - \frac{2}{3}\log\lambda + \mathfrak{f}_0\,,\\
\frac{F_2(\lambda)}{1-\mathfrak{g}} &= \frac{\hat{\lambda}^{-1}}{12\pi^2} - \frac{5\hat{\lambda}^{-\frac{1}{2}}}{36\sqrt{2}\pi} + \frac{2}{45}\,,\\
\frac{F_3(\lambda)}{1-\mathfrak{g}} &= \frac{\hat{\lambda}^{-2}}{144\pi^4} - \frac{\hat{\lambda}^{-\frac{3}{2}}}{162\sqrt{2}\pi^3} + \frac{19}{5670}\,.
\end{aligned}$$

where $\hat{\lambda} = \lambda - \frac{1}{24}$.

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How can we derive this from type IIA string theory?

Holography and black holes

The universal black hole

The following is a supersymmetric Euclidean solution of 4d $\mathcal{N}=2$ gauged supergravity [Romans], [Benetti Genolini-Ipiña-Sparks], [NPB-Charles-Min], ...

$$ds_4^2 = U(r)d\tau^2 + \frac{dr^2}{U(r)} + r^2 ds^2_{\Sigma_\mathfrak{g}}\,, \qquad F = \frac{Q}{r^2}d\tau\wedge dr - \frac{\kappa}{g}\mathrm{vol}(\Sigma_\mathfrak{g})\,,$$

$$U(r) = \left(\sqrt{2}gr + \frac{\kappa}{2\sqrt{2}gr}\right)^2 - \frac{Q^2}{8r^2}\,, \qquad \kappa = 1, 0, -1.$$

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The smooth Lorentzian black hole is obtained by taking $Q \to 0$ and exists only for $\kappa = -1$, i.e. $\mathfrak{g} > 1$. The regularized on-shell action for any β_τ is

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This is the extremal magnetic Reissner-Nordström black hole in AdS$_4$!

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An 11d uplift of this solution is given by

$$
\begin{aligned}
ds_{11}^2 &= \tfrac{1}{4} ds_4^2 + ds^2_{\mathbb{CP}^3} + \left(d\psi + \sigma_{\mathbb{CP}^3} + \tfrac{1}{4} A \right)^2 \ , \\
G_4 &= \tfrac{3}{8} \mathrm{vol}_4 - \tfrac{1}{4} \star_4 F \wedge J_{\mathbb{CP}^3} \ , \quad \text{with} \quad d\sigma_{\mathbb{CP}^3} = 2 J_{\mathbb{CP}^3} \ .
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Our QFT result for the TTI amounts to a prediction for the path integral of M-theory on this background to all orders in the $1/N$ expansion.

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An all-order prediction for the entropy of this black hole?!?

General "black saddles"

There are more general supersymmetric Euclidean "black saddle" solutions in the STU model of 4d $\mathcal{N} = 2$ supergravity (gravity + 3 vector multiplets)

$$ds_4^2 = e^{2f_1(r)}d\tau^2 + e^{2f_2(r)}dr^2 + e^{2f_3(r)}ds^2_{\Sigma_\mathfrak{g}}\,, \qquad A^I = v^I(r)d\tau + p^I\omega_{\Sigma_\mathfrak{g}}$$

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The on-shell action of these solutions agrees with the $N^{\frac{3}{2}}$ term in the TTI for general fugacities Δ_a and magnetic fluxes $\mathfrak{n}_a$. [NPB-Charles-Min]

$$F_{S^1\times\Sigma_\mathfrak{g}} \approx \frac{\pi\sqrt{2k\Delta_1\Delta_2\Delta_3\Delta_4}}{3}\sum_{a=1}^{4}\frac{\mathfrak{n}_a}{\Delta_a}N^{\frac{3}{2}}\,.$$

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Some of these Euclidean solutions admit Lorentzian interpretation as supersymmetric black holes with an $\mathrm{AdS}_2\times\Sigma_\mathfrak{g}$ near horizon limit. [Gauntlett-Kim-Pakis-Waldram], [Cacciatori-Klemm], [NPB-Min-Pilch], [NPB-Charles-Min]

The entropy of these black holes is computed by the TTI after a Legendre transform and $\mathcal{I}$-extremization. [Benini-Hristov-Zaffaroni], [NPB-Min-Pilch], ...

Other 3d $\mathcal{N} = 2$ SCFTs

3d $\mathcal{N} = 4$ SYM

There is another simple 3d $\mathcal{N} = 4$ holographic SCFT we can study with these tools - 3d $\mathcal{N} = 4$ SYM with $\mathrm{U}(N)$ gauge theory with 1 adjoint and N_f fundamental hypermultiplets and no CS term.

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The bulk dual is an $\mathrm{AdS}_4 \times S^7/\mathbb{Z}_{N_f}$ solution of 11d supergravity. The orbifold acts on $\mathbb{C}^4$ as

$$(z_1, z_2, z_3, z_4) \to (z_1, z_2, e^{2\pi \mathrm{i}/N_f} z_3, e^{2\pi \mathrm{i}/N_f} z_4)\,.$$

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We can apply the same numerical method to compute the TTI to find

$$\frac{F_{S^1 \times \Sigma_\mathfrak{g}}}{1 - \mathfrak{g}} = \frac{\pi\sqrt{2}}{3} N_f^{\frac{1}{2}} \left[\hat{N}^{\frac{3}{2}} - \left(\frac{N_f}{2} + \frac{5}{2N_f} \right) \hat{N}^{\frac{1}{2}} \right] + \frac{1}{2} \log \hat{N} - \hat{f}_0(N_f) \,,$$

where

$$\hat{N} = N + \frac{7N_f}{24} + \frac{1}{3N_f} \,.$$

There is a corresponding BPS black hole in M-theory for which this index computes the entropy.

Other examples

Similar results can be obtained for other 3d holographic SCFTs (no known Airy function on S^3 for most of these!)

- 3d $\mathcal{N}=2$ mABJM theory. We have $\hat{N} = N + \frac{19}{24}$

$$\frac{F_{S^1\times\Sigma_{\mathfrak{g}}}}{1-\mathfrak{g}} = \frac{4\pi\sqrt{2}}{9\sqrt{3}}\left[\hat{N}^{\frac{3}{2}} - \frac{9}{2}\hat{N}^{\frac{1}{2}}\right] + \frac{1}{2}\log\hat{N} - \hat{f}_0\,,$$

- 3d $\mathcal{N}=2$ $V^{5,2}$ theory. We have $\hat{N} = N + \frac{k}{6} + \frac{1}{4k}$ and

$$\frac{F_{S^1\times\Sigma_{\mathfrak{g}}}}{1-\mathfrak{g}} = \frac{16\pi\sqrt{k}}{27}\left[\hat{N}^{\frac{3}{2}} - \left(\frac{9k}{16} + \frac{27}{16k}\right)\hat{N}^{\frac{1}{2}}\right] + \frac{1}{2}\log\hat{N} - \hat{f}_0(k)\,,$$

- 3d $\mathcal{N}=2$ Q^{111} theory. We have $\hat{N} = N + \frac{k}{6}$ and

$$\frac{F_{S^1\times\Sigma_{\mathfrak{g}}}}{1-\mathfrak{g}} = \frac{4\pi\sqrt{k}}{3\sqrt{3}}\left[\hat{N}^{\frac{3}{2}} - \left(\frac{k}{4} + \frac{3}{4k}\right)\hat{N}^{\frac{1}{2}}\right] + \frac{1}{2}\log\hat{N} - \hat{f}_0(k)\,,$$

- 3d $\mathcal{N}=3$ N^{010} theory. We have $\hat{N} = N + \frac{k}{12} + \frac{1}{3k}$ and

$$\frac{F_{S^1\times\Sigma_{\mathfrak{g}}}}{1-\mathfrak{g}} = \frac{4\pi\sqrt{k}}{3\sqrt{3}}\left[\hat{N}^{\frac{3}{2}} - \left(\frac{k}{4} + \frac{5}{4k}\right)\hat{N}^{\frac{1}{2}}\right] + \frac{1}{2}\log\hat{N} - \hat{f}_0(k)\,,$$

In each of these cases there is a BPS black hole solution for which the index accounts for the entropy.

The superconformal index [with Sunjin Choi]

The superconformal index (SCI), or $S^1 \times_w S^2$ partition function, counts certain BPS operators in 3d $\mathcal{N} = 2$ SCFTs.

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For the ABJM theory in the M-theory limit we find the following ω^{-1} and ω^0 results

$$\begin{aligned}
&\log \mathcal{I}_{\mathsf{ABJM}}(N, k, \omega) \\
&= -\frac{\pi\sqrt{2k}}{3}\left[\left(\frac{1}{2\omega} + 1\right)\left(N - \frac{k}{24} + \frac{2}{3k}\right)^{\frac{3}{2}} - \frac{3}{k}\left(N - \frac{k}{24} + \frac{2}{3k}\right)^{\frac{1}{2}}\right] \\
&\quad - \frac{2}{\omega}\hat{g}_0(k) - \frac{1}{2}\log\left(N - \frac{k}{24} + \frac{2}{3k}\right) + \hat{f}_0(k) + \mathcal{O}(e^{-\sqrt{N}}) + \mathcal{O}(\omega)\,.
\end{aligned}$$

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The superconformal index (SCI), or $S^1 \times_w S^2$ partition function, counts certain BPS operators in 3d $\mathcal{N} = 2$ SCFTs.

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For the ABJM theory in the M-theory limit we find the following ω^{-1} and ω^0 results

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Similar results for other 3d $\mathcal{N} = 2$ holographic SCFTs.

Thermal observables

Consider a 3d CFT on $S^1_\beta \times \mathbb{R}^2$ where $\beta = 1/T$. The 1pt function of the energy-momentum tensor and the thermal free energy are

$$\langle \mathcal{T}^{00} \rangle = \frac{2}{3}\frac{b_{\mathcal{T}}}{\beta^3}\,, \qquad F_{S^1_\beta \times \mathbb{R}^2} = \frac{f_{\mathcal{T}}}{\beta^3}\,, \qquad 3 f_{\mathcal{T}} = b_{\mathcal{T}}\,.$$

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Similar results for other 3d $\mathcal{N}=2$ holographic SCFTs and for the thermal path integral on $S^1 \times S^2$.

Summary

- Presented exact results for the large N limit of the partition function of the ABJM theory on S^3, $S^1 \times \Sigma_{\mathfrak{g}}$, and $S^1 \times_\omega S^2$.
- Discussed how some of these results can be reproduced by string/M-theory via AdS/CFT.
- All order microscopic prediction for the entropy of the BPS AdS$_4$ Reissner-Nordström and Kerr-Newman black holes.
- Generalization of these results to some other 3d $\mathcal{N} = 2$, $\mathcal{N} = 3$, and $\mathcal{N} = 4$ holographic SCFTs.
- Applications of these results to the calculation of thermal observables.

Outlook

- Extend to other 3d $\mathcal{N} = 2$ holographic SCFTs. [in progress]
- Analytic derivation of our results.
- Partition functions on other compact 3-manifolds.
- Understand the shift in N from M-theory. [Bergman-Hirano], [in progress]
- Implications for the higher-derivative corrections to 4d and 11d supergravity?
- Supersymmetric localization in 4d supergravity?
- Derivation from (or lessons for) type IIA string theory and M-theory?
- OSV-type conjecture for AdS black holes?


thank you
danke
謝謝
gracias
merci
obrigado
teşekkür ederim
ngiyabonga
go raibh maith agat
bedankt
dziękuję
sukriya
kop khun krap
terima kasih
mochchakkeram
dank je
спасибо
arigatô
dakujem
grazie
tapadh leat
maururu
hvala
sagolun
taiku
takk
감사합니다
ありがとう
ευχαριστώ
мерси
рахмат
Баярлалаа
faafetai lava
kiitos
dankie
dhanyavad
nanni
nandri
vinaka
спасибі
blagodaram
mersi
kia ora
barka
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najis tuke
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diolch
dhanyavadagalu
shukriya
mercé
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